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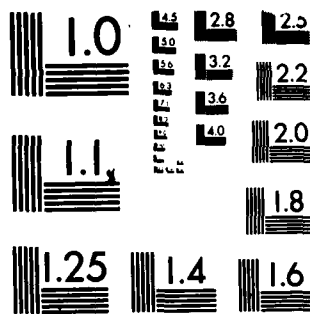
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REMARKS ON THE FLOW BETWEEN TWO PARALLEL
ROTATING PLATES

Seymour V. Parter and K. R. Rajagopal

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Seymour V. Parter and K. R. Rajagopal

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ABSTRACT

✓ The flow of the classical linearly viscous fluid between two infinite parallel planes rotating with constant (but different) angular velocities about a common axis has received a great deal of attention during the past 60 years (cf. Parter [12]). However, until recently the emphasis has been on solutions which are axi-symmetric. Recently, Berker [3] in his study of the flow between parallel planes rotating with the same constant angular velocities about a common axis exhibited a one parameter family of solutions only one of which is axi-symmetric. In this study, ^{the authors} we exhibit the existence of a one parameter family of solutions (for "large" viscosities) when the planes are rotating with constant but different angular velocities about a common axis or about non-coincident axes.

AMS (MOS) Subject Classifications: 34B15, 34D10, 76D05

Key Words: Ordinary differential equation, rotating fluids, non-uniqueness

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

The flow between parallel planes rotating with constant (but different) angular velocities about a common axis was first studied by Batchelor [2] instigated by an earlier study of von Karman [9] in 1923 and it has been the object of considerable analysis ever since then. The Karman assumptions, which has always been employed in studying the problem, leads to solutions which are axi-symmetric. In this study we show that the classical solutions are imbedded in a one parameter family of solutions which are basically non-symmetric and hence the classical solutions are not "isolated" or "stable". These results which seem to arise due to the unboundedness of the flow domain lead one to question the meaning and validity of the study of the above boundary value problem. Similar one parameter family of solutions are shown to exist in the case of the flow of rotating planes of constant but differing angular velocities about non-coincident axes.



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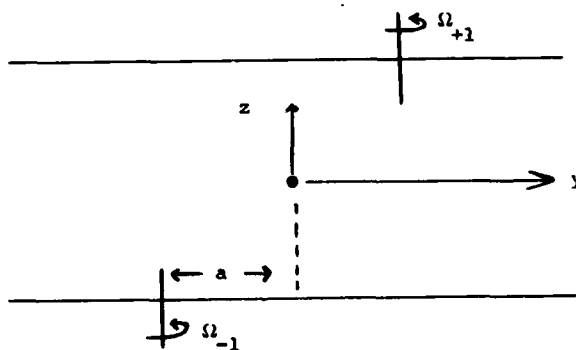
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REMARKS ON THE FLOW BETWEEN TWO
PARALLEL ROTATING PLATES

Seymour V. Parter and K. R. Rajagopal

Introduction

Let Π_1 and Π_2 be two infinite planes parallel to the (x,y) plane, say Π_1 is the plane $z = -1$ while Π_2 is the plane $z = 1$. Let $a > 0$ be a fixed constant and suppose that Π_1 rotates about a point $(x = 0, y = -a/2, z = -1)$ with constant angular velocity Ω_{-1} while Π_2 rotates about the point $(x = 0, y = +a/2, z = 1)$ with constant angular velocity Ω_{+1} . We suppose a classical incompressible fluid fills the infinite space between these planes and we seek steady state solutions of the Navier-Stokes equation which describe the fluid flow.



Finally, we make the basic assumption that

$$1.1) \quad U_z = -H(z),$$

that is, the component of velocity in the z direction is a function of z alone.

If $a = 0$ and we also assume that the flow is axi-symmetric then the basic theory of von Karman [9] and Batchelor [2] leads to the following conclusions:

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$$1.2) \quad U_r = \frac{\varepsilon}{2} H'(z) \quad r = (x^2 + y^2)^{1/2}$$

$$1.3) \quad U_\theta = \frac{\varepsilon}{2} G(z)$$

where the functions $\langle G(z), H(z) \rangle$ are solutions of the boundary-value problem

$$1.4a) \quad \varepsilon H^{(4)} + HH''' + GG' = 0, \quad -1 < z < 1$$

$$1.4b) \quad \varepsilon G'' + HG' - H'G = 0, \quad -1 < z < 1$$

and

$$1.5a) \quad H(-1, \varepsilon) = H(1, \varepsilon) = 0 \quad (\text{no penetration})$$

$$1.5b) \quad H'(-1, \varepsilon) = H'(1, \varepsilon) = 0 \quad (\text{no slip})$$

$$1.5c) \quad G(-1, \varepsilon) = 2\Omega_{-1}, \quad G(1, \varepsilon) = 2\Omega_{+1}$$

where the positive parameter $\varepsilon > 0$ is related to the bulk viscosity

This boundary value problem has been studied at great length. There are many numerical studies and many formal asymptotic studies. There are also rigorous existence theorems

- (i) for $\varepsilon \gg 1$ by Hastings [8] and Elcrat [6],
- (ii) for $\Omega_{-1} = -\Omega_{+1} \neq 0$ and all $\varepsilon > 0$ by McLeod and Parter [11],
- (iii) for $0 < \varepsilon \ll 1$ by Kreiss and Parter [10].

The recent survey article [12] contains a reasonably up-to-date discussion of this problem.

When $\Omega_{-1} = \Omega_{+1} \neq 0$ there is one special solution (not the only solution—see [3]): the rigid body rotation given by

$$1.6) \quad H(z, \varepsilon) \equiv 0, \quad G(z, \varepsilon) \equiv 2\Omega_{-1}.$$

It is not difficult to verify ([5], [18]) that this solution is "stable" and "isolated" relative to the von Karman equations (1.4), (1.5). By "isolated" we mean there is a neighborhood of this solution wherein there are no other solutions, and by "stable" we mean there is no bifurcation from this solution, in particular the linearized problem at this solution is non-singular. On the other hand, Berker [3] has constructed a one parameter family of solutions to the general steady Navier-Stokes equation which includes the rigid body motion. The rigid body motion is the only axisymmetric solution that

belongs to Berker's [3] family. In Cartesian coordinates his solution takes the form (we have set $\Omega_{-1} = \Omega_{+1} = 1$).

$$1.7) \quad U_x = -[y-g(z)], \quad U_y = [x-f(z)], \quad U_z = 0$$

where

$$1.8a) \quad f(z) = l \left\{ \frac{1-\phi(1)}{\Delta} [\phi(z)-\phi(a)] - \frac{\chi(1)}{\Delta} [\chi(z)-\chi(1)] \right\},$$

$$1.8b) \quad g(z) = l \left\{ \frac{\chi(1)}{\Delta} [\phi(z)-\phi(1)] + \frac{1-\phi(1)}{\Delta} [\chi(z)-\chi(1)] \right\},$$

with

$$1.9a) \quad \phi(z) = \cosh mz \cdot \cos mz,$$

$$1.9b) \quad \chi(z) = \sinh mz \cdot \sin mz,$$

$$1.9c) \quad m = \left(\frac{1}{2\epsilon}\right)^{1/2}$$

$$1.9d) \quad \Delta = [1-\phi(1)]^2 + [\chi(1)]^2 = (\cosh m - \cos m)^2$$

while l is an arbitrary positive constant. Observe that (1.7) shows that this solution satisfies the basic ansatz (1.1).

The case $a \neq 0$ and $\Omega_{-1} = \Omega_{+1} = \Omega$ has relevance to the flow occurring in the orthogonal rheometer, an instrument that is employed in determining the material moduli which characterize non-Newtonian fluids. Recently, Rajagopal [13] has studied the flow of general simple fluids in such a domain and Rajagopal and Gupta [15], and Rajagopal and Wineman [17] have established exact solutions to the problem for certain non-Newtonian fluids. Rajagopal and Gupta [16] have also established a one parameter family of exact solution for an incompressible homogeneous fluid of second grade when $a = 0$ and $\Omega_{-1} = \Omega_{+1} = \Omega$.

When $a = 0$ and $\Omega_{-1} = \Omega_{+1}$, the solution corresponding to the usual Karman ansatz leads to exactly one of the solutions in Berker's [3] one parameter family, namely the rigid motion which is axi-symmetric. In the case $a \neq 0$, and $\Omega_{-1} = \Omega_{+1}$, an exact solution has been obtained for the classical incompressible fluid by Abbot and Walters

More recently, Goddard [7] has also established results which are similar to those in [13].

[1]. The existence of such a solution is implied in an earlier analysis of Berker [4]. This motivates us to look for a more general class of solutions to the Karman problem (and to the corresponding problem when $a \neq 0$) which would reduce to the class of non-axisymmetric solutions exhibited by Berker [3]. Thus, we seek a solution - for classical incompressible fluids - which, in Cartesian coordinates - take the form

$$1.10a) \quad U_x = \frac{x}{2} H'(z) - \frac{G(z)}{2} y + g(z)$$

$$1.10b) \quad U_y = \frac{y}{2} H'(z) + \frac{G(z)}{2} x - f(z)$$

$$1.10c) \quad U_z = -H(z).$$

In cylindrical coordinates this velocity field takes the form

$$1.11a) \quad V_r = \frac{x}{2} H'(z) + g(z) \cos \theta - f(z) \sin \theta,$$

$$1.11b) \quad V_\theta = \frac{x}{2} G(z) - g(z) \sin \theta - f(z) \cos \theta,$$

$$1.11c) \quad V_z = -H(z).$$

Observe that, if $H = 0$ and $G = 2\Omega$ we have a velocity field of the form described by Berker. And, if $f = g = 0$ we have a velocity field of the form described by von Karman.

As we show in section 2, there is a solution of the steady state Navier-Stokes equations of the form (1.10) [or (1.11)] if and only if the functions $H(z)$, $G(z)$, $g(z)$, $f(z)$ are solutions of the boundary-value problem

$$1.12a) \quad cH^{iv} + HH'''' + GG' = 0, \quad -1 < z < 1$$

$$1.12b) \quad cG'' + HG' - H'G = 0, \quad -1 < z < 1$$

$$1.13a) \quad cf'''' + Hf'' + \frac{1}{2}H'f' - \frac{1}{2}H''f + \frac{1}{2}(Gg)' = 0$$

$$1.13b) \quad cg'''' + Hg'' + \frac{1}{2}H'g' - \frac{1}{2}H''g - \frac{1}{2}(Gf)' = 0$$

$$1.14a) \quad H(-1, \epsilon) = H(1, \epsilon) = 0, \quad (\text{no penetration}),$$

$$1.14b) \quad H'(-1, \epsilon) = H'(1, \epsilon) = 0, \quad (\text{no slip}),$$

$$1.14c) \quad G(-1, \epsilon) = 2\Omega_{-1}, \quad G(1, \epsilon) = 2\Omega_{+1}$$

$$1.15) \quad f(-1, \epsilon) = f(1, \epsilon) = 0, \quad g(-1, \epsilon) = -a\Omega_{-1}/2, \quad g(1, \epsilon) = a\Omega_{+1}/2.$$

We note that equations (1.12) with boundary conditions (1.14) are exactly the nonlinear von Karman equations for axially symmetric swirling flow for functions

$\langle H(z, \epsilon), G(z, \epsilon) \rangle$ while the equations (1.13) with boundary conditions (1.15) are linear

equations for $\langle f(z, \epsilon), g(z, \epsilon) \rangle$ with coefficients depending on $\langle H(z, \epsilon), G(z, \epsilon) \rangle$ which reflect the asymmetry and the possible displacement of the centers of rotation of the bounding planes. Moreover, given $\langle H(z, \epsilon), G(z, \epsilon) \rangle$ the of two third order equations with only four boundary conditions. Whenever, there is a solution of the von Karman equations one can ask two questions:

(i) In the case when $a = 0$, is the axi-symmetric flow imbedded in a continuous one parameter family of more general solutions?

(ii) When $a \neq 0$, does this axi-symmetric flow form the basis for a one parameter family of solutions of the problem for rotations about different centers?

In case (i) this is a homogeneous underdetermined system and the answer is yes! We need merely consider the additional condition

$$1.16) \quad f'(-1) = g'(-1) = 0.$$

If this augmented homogeneous problem has a non-trivial solution

$\langle f(z, \epsilon), g(z, \epsilon) \rangle \neq \langle 0, 0 \rangle$, then $\langle lf(z, \epsilon), lg(z, \epsilon) \rangle$ is also a solution for every real number l . On the other hand, if the system (1.13), (1.15), (1.16) does not have a nontrivial solution; then the problem given by (1.13), (1.15) and

$$1.17) \quad f'(-1) = l, \quad g'(-1) = 0$$

yields a unique solution $\langle f(z, \epsilon, l), g(z, \epsilon, l) \rangle$ of the form

$$\langle f(z, \epsilon, l), g(z, \epsilon, l) \rangle = \langle lf(z, \epsilon, 1), lg(z, \epsilon, 1) \rangle.$$

This simple result has the following important consequence. In the classical case of two infinite parallel planes rotating about a common axis, (i.e., $a = 0$) whenever there is a solution of the von Karman equations (12), (14), this axi-symmetric flow is imbedded in a one-parameter family of solutions of the full Navier-Stokes equations. Thus, despite the intense interest in the von Karman problem within the class of all solutions of the Navier-Stokes equations, these special solutions are "unstable". While it seems likely that a similar simple argument settles the matter for case (ii), it is not apparent.

In either case one can ask a more subtle question: can we find a family $\langle f(z, \epsilon, l), g(z, \epsilon, l) \rangle$ which is continuous in both ϵ and l and (at the same time) has

the geometric significance of the Berker solution for the special case $\Omega_{-1} = \Omega_{+1} \neq 0$?

In other words, can we find solutions of (1.12), (1.13), (1.14), (1.15) and

$$(1.18) \quad g(0, \epsilon, l) = 0, \quad f(0, \epsilon, l) = l.$$

Since the system (1.13), (1.15) is linear, the answer is 'yes' for problem (i) if and only if it is also 'yes' for problem (ii).

In section 3 we answer these questions in the affirmative for large ϵ . While this result is an immediate consequence of the implicit function theorem (applied at $\epsilon = \infty$) we will give a complete proof.

2. Equations of Motion

In this discussion we follow the outline of the argument given in [12, section 2]. A velocity field of the form (1.10) satisfies

$$(2.1) \quad \operatorname{div} U = \frac{1}{2} H'(z) + \frac{1}{2} H'(z) - H'(z) = 0.$$

Thus, the basic constraint of the Navier-Stokes equations is satisfied. We now turn to the equation

$$(2.2) \quad \mu \Delta u - \rho u \cdot \nabla u = \nabla p$$

where μ denotes the viscosity, ρ the density, and p is the pressure. We eliminate the pressure by taking the curl of both sides of (2.2) and obtain

$$(2.3) \quad \mu \Delta \omega - \rho (\omega \cdot \nabla) U = 0$$

where

$$(2.3b) \quad \omega = \operatorname{curl} U.$$

A detailed calculation now yields

$$(2.4) \quad \Delta \omega = -\frac{1}{2} \left\{ \frac{Y}{2} H^{iv} + \frac{X}{2} G^{''''} - \frac{f^{''''}}{2} \right\} + \frac{1}{2} \left\{ \frac{X}{2} H^{iv} - \frac{Y}{2} G^{''''} + \frac{g^{''''}}{2} \right\} + k (G'')$$

while

$$\omega \times U = -\frac{1}{2} \left\{ \frac{X}{2} [(GH')' - (G'H)'] - \frac{Y}{2} [GG' + HH''] \right\} + \left[\frac{(Gg)'}{4} - \frac{(fH')'}{4} + \frac{(f'H)'}{2} \right]$$

$$2.5) \quad + \frac{1}{2} \left\{ -\frac{x}{2} [GG' + HH'''] + \frac{y}{2} [(G'H)' - (GH')'] + \left[\frac{(Gf)'}{4} + \frac{(gH')'}{4} - \frac{(g'H)'}{2} \right] \right\} \\ + \frac{k}{2} \{ GH' - HG' \} .$$

On equating the coefficients of $\frac{k}{2}$ in (2.3a) we obtain

$$2.6a) \quad \frac{\mu}{\rho} G'' + HG' - H'G = 0.$$

On equating the coefficients of $\frac{1}{2}x$ we obtain

$$\frac{\mu}{\rho} G''' + (HG')' - (H'G)' = 0$$

which also follows from (2.6a). On equating the coefficients of $\frac{1}{2}y$ we obtain

$$2.6b) \quad \frac{\mu}{\rho} H^{iv} + HH''' + GG' = 0.$$

Finally, the zeroth order terms in the coefficient of $\frac{1}{2}$ yield

$$2.6c) \quad \frac{\mu}{\rho} f''' + (Hf')' - \frac{1}{2}(H'f)' + \frac{1}{2}(Gg)' = 0.$$

The coefficients of $\frac{1}{2}x$, $\frac{1}{2}y$ and $\frac{1}{2}$ yield the final equation

$$2.6d) \quad \frac{\mu}{\rho} g''' + (Hg')' - \frac{1}{2}(H'g)' + \frac{1}{2}(Gf)' = 0.$$

Thus we have established the equations (1.12) and (1.13) with

$$\epsilon = \frac{\mu}{\rho}.$$

We now turn to the boundary conditions. For our immediate purposes it would be sufficient to impose the conditions (1.4) and (1.5) and observe that the velocity field (1.10) now satisfies the steady-state Navier-Stokes equations and the boundary conditions of our problem. However, it is somewhat more satisfying to proceed as follows. Equation (1.11c) and the "no penetration" condition implies that

$$2.7a) \quad H(-1, \epsilon) = H(1, \epsilon) = 0$$

Equation (1.11b) and the boundary conditions

$$V_\theta(r, \theta, \pm 1) = (r \pm \frac{3}{2}) \Omega_{\pm 1}$$

yield

$$\lim_{r \rightarrow \infty} \frac{1}{r} V_\theta(r, \theta, \pm 1) = \Omega_{\pm 1} = \frac{1}{2} G(\pm 1).$$

Hence

$$2.7b) \quad G(-1, \epsilon) = 2\Omega_{-1}, \quad G(1, \epsilon) = 2\Omega_{+1}.$$

thus, we have obtained the boundary conditions (1.14). Now, using (1.11b) and letting $r \rightarrow 0$ with a judicious choice of θ we obtain the boundary conditions (1.15).

3. Existence for large $\epsilon \gg 1$

In this section we present what is essentially a standard argument for regular perturbation problems. The argument is given in some detail because we wish to emphasize the following facts.

(i) There is an $\epsilon_0 \gg 1$ and, for all $\epsilon > \epsilon_0$ there is a solution of the von Karman problem (1.4), (1.5). Moreover, this solution is continuous in ϵ . Hence, there is a curve of solutions and there is no local bifurcation of solutions of the von Karman equation from this curve. Again, within the set of solutions of the von Karman equations, for fixed $\epsilon > \epsilon_0$, each of these solutions is isolated.

(ii) Nevertheless, in this same range of ϵ there is a solution of the full system (1.12)-(1.15): a one parameter family of solutions $\langle H(x, \epsilon), G(x, \epsilon), f(x, \epsilon), x, \epsilon, l \rangle$ which includes (for $l = 0$) the axi-symmetric von Karman solution. Moreover, if ϵ and l are both fixed, this solution is an isolated stable solution. Of course, with $a > 0$ these solutions provide a one parameter family of solutions of the problem of rotation about different centers.

Our first goal is to show that in the case of the von Karman equations a relatively simple Picard iteration scheme converges for $\epsilon \gg 1$ and - in the nature of things, the solutions so obtained are continuous in $R = 1/\epsilon$.

Definition: Let $f \in C^k[-1, 1]$, $k > 0$. Then

$$\|f\|_k = \sum_{j=0}^k \max(|f^{(j)}(x)| : -1 \leq x \leq 1)$$

Lemma 3.1: Consider the two boundary value problems:

$$3.1a) \quad \theta^{(4)} = f, \quad -1 \leq x \leq 1$$

$$3.1b) \quad \theta(\pm 1) = \theta'(\pm 1) = 0,$$

$$3.2a) \quad \psi'' = g, \quad -1 < x < 1,$$

$$3.2b) \quad \psi(-1) = 2\Omega_{-1}, \quad \psi(1) = 2\Omega_{+1}.$$

There is a constant $K_0 > 1$ such that

$$3.3a) \quad \|\theta\|_4 < K_0 \|\theta\|_0$$

$$3.3b) \quad \|\psi\|_2 < K_0 [\|\theta\|_0 + 2|\Omega_{-1}| + 2|\Omega_{+1}|]$$

Proof: Direct integration.

Let Ω_{-1} and Ω_{+1} be given. Let

$$3.4a) \quad \sigma = 2K_0 [|\Omega_{-1}| + |\Omega_{+1}| + 1],$$

$$3.4b) \quad R = \frac{1}{\epsilon} < R_0 = \frac{1}{16K_0\sigma}.$$

Let

$$3.5a) \quad H_0 \equiv 0,$$

$$3.5b) \quad G_0 = 2 \left[\Omega_{-1} + \frac{(x+1)}{2} (\Omega_{+1} - \Omega_{-1}) \right]$$

and consider the iterative scheme

$$3.6a) \quad H_{k+1}^{IV} = -R[H_k H_k''' + G_k G_k']$$

$$3.6b) \quad H_{k+1}(\pm 1) = H_{k+1}'(\pm 1) = 0$$

$$3.7a) \quad G_{k+1}'' = R[H_k G_k - H_k G_k']$$

$$3.7b) \quad G_{k+1}(-1) = 2\Omega_{-1}, \quad G_{k+1}(1) = 2\Omega_{+1}$$

Lemma 3.2: If

$$3.8a) \quad \|H_k\|_4 < \sigma, \quad \|G_k\|_2 < \sigma.$$

Then

$$3.8b) \quad \|H_{k+1}\|_4 < \sigma \quad \text{and} \quad \|G_{k+1}\|_2 < \sigma.$$

Proof: From the definition of K_0 we have

$$3.9a) \quad \|H_{k+1}\|_4 < 2K_0 R \sigma^2 < \frac{2K_0}{16K_0\sigma} \cdot \sigma^2 = \frac{1}{8} \sigma < \sigma$$

and

$$\begin{aligned} \|G_{k+1}\|_2 &< K_0 [2R\sigma^2 + 2|\Omega_{-1}| + 2|\Omega_{+1}|] \\ &< \frac{2K_0}{16K_0\sigma} \sigma^2 + 2K_0[|\Omega_{-1}| + |\Omega_{+1}|]. \end{aligned}$$

That is

$$3.9b) \quad \|G_{k+1}\|_2 < \frac{\sigma}{8} + \frac{\sigma}{10} < \sigma.$$

Lemma 3.3: Suppose

$$3.10) \quad \|H_k\|_4 < \sigma, \|G_k\|_2 < \sigma, k = 0, 1, \dots$$

Then, for $k > 1$

$$3.11) \quad (\|H_{k+1} - H_k\|_4 + \|G_{k+1} - G_k\|_2) < \frac{1}{4} (\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2)$$

Proof:

$$\begin{aligned} (H_{k+1} - H_k)^{IV} &= -R \{ (H_k - H_{k-1}) H_k''' + H_{k-1} (H_k''') \\ &\quad + G_k' (G_k - G_{k-1}) + G_{k-1} (G_k' - G_{k-1}') \} \\ \|H_{k+1} - H_k\|_4^4 &< 2K_0 R_0 - \{ \|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2 \} \\ &= \frac{1}{8} (\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2). \end{aligned}$$

Thus

$$(3.12a) \quad \|H_{k+1} - H_k\|_4^4 < \frac{1}{4} (\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2).$$

And

$$\begin{aligned} (G_{k+1} - G_k)'' &= R \{ (H_k' - H_{k-1}') G_k + (G_k - G_{k-1}) H_{k-1}' \\ &\quad + (H_k - H_{k-1}) G_k' - (G_k' - G_{k-1}') H_{k-1} \}. \end{aligned}$$

Hence

$$\begin{aligned} 3.12b) \quad \|G_{k+1} - G_k\|_2 &< 2K_0 R_0 [\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2] \\ &< \frac{1}{8} (\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2). \end{aligned}$$

Adding (3.12a) and (3.12b) gives the desired result

Theorem 3.1: If $R = \frac{1}{\epsilon} < R_0 = \frac{1}{16K_0 c}$

$$\sigma = 20 K_0 (|\Omega_{-1}| + |\Omega_{+1}|) + 1.$$

Then the iterative procedure (3.5)-(3.7) converges to an isolated solution

$\langle H(x, \epsilon), G(x, \epsilon) \rangle$ which is continuous in ϵ for $\epsilon > \frac{1}{R_0}$.

Proof: The proof is now a standard argument based on the estimates of lemma 3.2 and lemma 3.3.

We now turn to the linear equations (1.13) with boundary conditions (1.15).

Lemma 3.4: Consider the multi-point problem

$$3.13a) \quad v''' = F, \quad -1 < x < 1,$$

$$3.13b) \quad v(-1) = A, \quad v(1) = B, \quad v(0) = c.$$

Let (x) be the triple integral of F , i.e.,

$$(x) = \int_{-1}^x dy \int_{-1}^y dt \int_{-1}^t F(s) ds.$$

Then, the solution of (3.13a), (3.13b) is given by

$$3.14a) \quad v(x) = A + \alpha(x+1) + \frac{\beta}{2}(x+1)^2 + (x)$$

where

$$3.14b) \quad \alpha = 2c - \frac{3}{2}A - \frac{1}{2}B - 2(0) + \frac{1}{2}(1)$$

$$3.14c) \quad \beta = (B+A-2C) + 2(0) - (1)$$

Proof: direct verification.

Corollary 3.4: There is a constant K_1 such that

$$3.15) \quad |v| < K_1 [|A| + |B| + |C| + |F|_0].$$

Given $\langle H(x, \epsilon), G(x, \epsilon) \rangle$ for $\epsilon > \epsilon_0$ let us consider the iterative procedure

$$3.16a) \quad f_{k+1}''' = 4R[-(Gg_k)'' + \frac{1}{2}H''f_k - Hf_k'' - \frac{1}{2}H'f_k'],$$

$$3.16b) \quad f(-1) = 0, \quad f(0) = 1, \quad f(1) = 0$$

$$3.17a) \quad g_{k+1}' = R[(Gf_k)' + \frac{1}{2}H''g_k - Hg_k'' - \frac{1}{2}H'g_k']$$

$$3.18a) \quad g(-1) = -a\Omega_{-1}/2, \quad g(0) = 0, \quad g(1) = a\Omega_{+1}/2.$$

Quite clearly, the arguments above show that there is an

$\epsilon_1 > \epsilon_0$ and, for all $\epsilon > \epsilon_1 > \epsilon_0$ this multi-point problem possesses a unique solution

which is continuous in ϵ and δ . Thus we have verified all the opening remarks of this section.

4. Remarks

We conclude this analysis by making a few observations on the significance of the result established in the previous section.

We have studied special solutions of the Navier-Stokes equations for a fluid contained with two infinite parallel planes each rotating with a constant angular velocity Ω_k ($k = \pm 1$). The centers of rotation may or may not lie on the same axis perpendicular to the planes.

In either case we are led to a system of ordinary differential equations which contain (as a subset) the classical equations of von Karman [9] and Batchelor [2] for special axi-symmetric flow about a common axis. In particular, in this classical case studied by von Karman and Batchelor, if there are such special solutions, they are never isolated solutions when considered with the scope of the full Navier-Stokes equations. In the case of "off-centered" rotation there are many unanswered questions. However, we have shown that (contrary to most intuitive ideas) in the case of "large" viscosity, there are solutions and they are never isolated. While the underlying basis for these anomalies is not completely understood, we believe it is related to the fact that in this unbounded domain the velocities at large r are great.

It is also worth observing that similar results can be established in the case of the flow of a Newtonian fluid between rotating porous disks and $\epsilon \gg 1$. In this case, the only change in the problem is in the boundary condition (1.14a). It is an easy matter to modify the arguments of Section 3.

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ABSTRACT (Continued)

of a one parameter family of solutions (for "large" viscosities) when the planes are rotating with constant but different angular velocities about a common axis or about non-coincident axes.

